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Asymptotic entanglement capacity of the Ising and anisotropic Heisenberg interactions

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We calculate the asymptotic entanglement capacity of the Ising interaction $\sigma_z \otimes \sigma_z$, the anisotropic Heisenberg interaction $\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y$, and more generally, any two-qubit Hamiltonian with canonical form $K = \mu_x\ \sigma_x \otimes \sigma_x + \mu_y\ \sigma_y \otimes \sigma_y$. We also describe an entanglement assisted classical communication protocol using the Hamiltonian K with rate equal to the asymptotic entanglement capacity.

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The fundamental resource for quantum information processing is an interaction between two quantum systems. Any Hamiltonian $H_{AB} \neq H_A + H_B$ that is not a sum of local terms couples the systems A and B. Together with local operations, the coupling can be used to generate entanglement [1, 2, 3], to transmit classical and quantum information [2, 4, 5, 6], and more generally, to simulate the bipartite dynamics of some other Hamiltonian H'_{AB} and thus to perform arbitrary unitary gates on the composite space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ [7, 8, 9].

Much experimental effort has been devoted to creating entangled states of quantum systems, including those in quantum optics, nuclear magnetic resonance, and condensed matter physics [10]. Determining the ability of a system to create entangled states provides a benchmark of the "quantumness" of the system. Furthermore, such states could ultimately be put to practical use in various quantum information processing tasks, such as superdense coding [11] or quantum teleportation [12].

The theory of optimal entanglement generation can be approached in different ways. For example, Ref. [1] considers *single-shot* capacities. In the case of two-qubit interactions, and assuming that ancillary systems are not available, Ref. [1] presents a closed form expression for the entanglement capacity and optimal protocols by which it can be achieved. In contrast, Ref. [2] considers the *asymptotic* entanglement capacity, allowing the use of ancillary systems, and shows that when ancillas are allowed, the single-shot and asymptotic capacities are in fact the same. However, such capacities can be difficult to calculate because the ancillary systems may be arbitrarily large.

In this paper, we calculate the asymptotic entanglement capacity of any two-qubit interaction that is locally equivalent to $\mu_x\ \sigma_x \otimes \sigma_x + \mu_y\ \sigma_y \otimes \sigma_y$, and thus present a connection between the results of Refs. [1] and [2]. We consider the use of ancillary systems, and show that they do not increase the entanglement capacity of these interactions. Thus in these cases, the asymptotic capacity discussed in Ref. [2] is in fact given by the expression presented in Ref. [1]. As an application of this result, we present an explicit ensemble for entanglement assisted classical communication [2], implicitly found in Ref. [6], at a rate equal to the entanglement capacity. We also give an alternative ensemble achieving the same rate. Finally, we conclude by presenting some numerical results on the entanglement capacity of general two-qubit interactions.

We begin by reviewing some definitions and known results. Let $|\psi\rangle$ be a state of the systems A and B. This state can always be written using the Schmidt decomposition [13],

$$|\psi\rangle := \sum_i \sqrt{\lambda_i}\ |\phi_i\rangle_A \otimes |\eta_i\rangle_B\,, \tag{1}$$

where $\{|\phi_i\rangle\}$ and $\{|\eta_i\rangle\}$ are orthonormal sets of states, and $\lambda_i > 0$ with $\sum_i \lambda_i = 1$. The entanglement between A and B is defined as

$$E(|\psi\rangle) := -\sum_i \lambda_i \log \lambda_i\,. \tag{2}$$

(Throughout this paper, the base of log is 2.)

Reference [1] considers maximizing the rate of increase of entanglement when a pure state is acted on by e^{-iHt}, the evolution according to a time-independent Hamiltonian H (we set $\hbar = 1$ throughout this paper). We refer to this maximal rate as the *single-shot* entanglement capacity. When no ancillas are used, this is given by

$$E_H^{(1*)} := \max_{|\psi\rangle \in \mathcal{H}_{AB}} \lim_{t\to 0} \frac{E(e^{-iHt}|\psi\rangle) - E(|\psi\rangle)}{t}\,. \tag{3}$$

Here the rate of increasing entanglement is optimized over all possible pure initial states of $\mathcal{H}_{AB}$ without ancillary systems. In fact, the single-shot capacity may be higher if ancillary systems A' and B', not acted on by H, are used. For this reason, we may consider the alternative single-shot entanglement capacity

$$E_H^{(1)} := \sup_{|\psi\rangle \in \mathcal{H}_{AA'BB'}} \lim_{t\to 0} \frac{E(e^{-iHt}|\psi\rangle) - E(|\psi\rangle)}{t} \,. \quad (4)$$

Note that in Eqs. (3) and (4), the limit is the same from both sides even though it might be the case that $E_H^{(1*)} \neq E_{-H}^{(1*)}$ in general (and similarly for $E_H^{(1)}$).

For any two-qubit Hamiltonian H, Ref. [1] shows that it is locally equivalent to a *canonical form*

$$\sum_{i=x,y,z} \mu_i \; \sigma_i \otimes \sigma_i \,, \quad \mu_x \geq \mu_y \geq |\mu_z| \,. \quad (5)$$

In terms of this canonical form, the optimal single-shot entanglement capacity of any two-qubit interaction without ancillas is given by

$$E_H^{(1*)} \;=\; \alpha(\mu_x + \mu_y) \,, \quad (6)$$

$$\alpha \;:=\; 2\max_x \sqrt{x\,(1-x)}\, \log\left(\frac{x}{1-x}\right) \approx 1.9123 \,, \quad (7)$$

where the maximum is obtained at $x_0 \approx 0.9168$. In addition, $E_H^{(1)}$ may be strictly larger than $E_H^{(1*)}$ when $|\mu_z| > 0$ [1].

Reference [2] considers the *asymptotic* entanglement capacity E_H for an arbitrary Hamiltonian H. E_H is defined as the maximum average rate at which entanglement can be produced by using many interacting pairs of systems, in parallel or sequentially. These systems may be acted on by arbitrary collective local operations (attaching or discarding ancillary systems, unitary transformations, and measurements). Furthermore, classical communication between A and B and possibly mixed initial states are allowed. Reference [2] proves that the asymptotic entanglement capacity in this general setting turns out to be just the single-shot capacity in Ref. [1], $E_H = E_H^{(1)}$, for all H, so

$$E_H = \sup_{|\psi\rangle \in \mathcal{H}_{AA'BB'}} \lim_{t\to 0} \frac{E(e^{-iHt}|\psi\rangle) - E(|\psi\rangle)}{t} \,. \quad (8)$$

Note that the definition of the capacity involves a supremum over both all possible states and all possible interaction times, but in fact it can be expressed as a supremum over states and a limit as $t \to 0$, with the limit and the supremum taken in either order.

Let $|\psi\rangle$ be the optimal input in Eq. (4) or (8). When $|\psi\rangle$ is finite dimensional, the entanglement capacity can be achieved [1, 2] by first inefficiently generating some EPR pairs, and repeating the following three steps: (i) transform $nE(|\psi\rangle)$ EPR pairs into $|\psi\rangle^{\otimes n}$ [14, 15], (ii) evolve each $|\psi\rangle$ according to H for a short time δt, and (iii) concentrate the entanglement into $n(E(|\psi\rangle)+\delta t E_H)$ EPR pairs [14].

In this paper, we show that $E_K^{(1)} = E_K^{(1*)}$ for any two-qubit Hamiltonian with canonical form

$$K := \mu_x \; \sigma_x \otimes \sigma_x + \mu_y \; \sigma_y \otimes \sigma_y \,, \quad \mu_x \geq \mu_y \geq 0 \,, \quad (9)$$

so that all three entanglement capacities are equal:

$$E_K = E_K^{(1)} = E_K^{(1*)} \,. \quad (10)$$

The optimal input is therefore a two-qubit state, and the optimal protocol applies. In particular, for these Hamiltonians, which include the Ising interaction $\sigma_z \otimes \sigma_z$ and the anisotropic Heisenberg interaction $\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y$, entanglement can be optimally generated from a 2-qubit initial state $|\psi\rangle$ without ancillary systems $A'B'$. As mentioned above, this result is not generic, since ancillas increase the amount of entanglement generated by some two-qubit interactions, such as the isotropic Heisenberg interaction $\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z$ [1].

In the following, we will focus on computing the asymptotic entanglement capacity of the interaction

$$K_{xx} := \sigma_x \otimes \sigma_x \,. \quad (11)$$

One way to see that this is sufficient to determine the asymptotic entanglement capacity of K in Eq. (9) is to note that K is *asymptotically equivalent* to

$$K' := (\mu_x + \mu_y)\sigma_x \otimes \sigma_x \quad (12)$$

and that $E_{tH} = |t|E_H$ for two-qubit Hamiltonians. The asymptotic equivalence of K and K' is based on the following two facts: (i) K' and fast local unitary transformations on qubits A and B can simulate K [8]; conversely, (ii) the Hamiltonian K can be used to simulate K' given a *catalytic* maximally entangled state, without consuming the entanglement of $A'B'$, which subsequently can be re-used [9]. Therefore, Hamiltonians K and K' are asymptotically equivalent resources given local operations and an asymptotically vanishing amount of entanglement. In particular, $E_K = E_{K'}$. This equivalence could be generalized to other capacities, but for the specific case of entanglement capacity, a simpler proof is available. The simulation (i), which does not require a catalyst, demonstrates $E_K \leq E_{K'}$. After computing $E_{K'}$, we will see that the protocol of Ref. [1] saturates this bound, so in fact $E_K = E_{K'}$ with no need for ancillas to achieve either capacity.

We now present the optimization of Eq. (8) for K_{xx}. We suppose that in addition to the qubits A and B on which K_{xx} acts, d-dimensional ancillas A' and B' are used, where d is arbitrary. We can always write the Schmidt-decomposed initial state $|\psi\rangle$ as

$$|\psi\rangle \;=\; \sum_i \sqrt{\lambda_i}\, |\phi_i\rangle_{AA'} \otimes |\eta_i\rangle_{BB'} \quad (13)$$

$$\;=\; (U \otimes V)(\sqrt{\Lambda} \otimes I_{BB'})|\Phi\rangle \quad (14)$$

$$\;=\; U\sqrt{\Lambda}V^T \otimes I_{BB'}|\Phi\rangle \,, \quad (15)$$

where U and V are unitary matrices on $\mathcal{H}_{AA'}$ and $\mathcal{H}_{BB'}$, Λ is a diagonal matrix with diagonal elements $\Lambda_{ii} = \lambda_i$, $|\Phi\rangle = \sum_i |i\rangle_{AA'} \otimes |i\rangle_{BB'}$, and we have used the fact that

$$(I \otimes M)|\Phi\rangle = (M^T \otimes I)|\Phi\rangle \tag{16}$$

for any operator M. Defining $\rho := \mathrm{tr}_{BB'} |\psi\rangle\langle\psi|$, the entanglement capacity of any Hamiltonian H is

$$\begin{aligned} E_H &= \sup_{|\psi\rangle} \mathrm{tr}\left(-\frac{\mathrm{d}\rho}{\mathrm{d}t}\log\rho - \rho\frac{\mathrm{d}\log\rho}{\mathrm{d}t}\right) \\ &= \sup_{|\psi\rangle} \mathrm{tr}\left(-\frac{\mathrm{d}\rho}{\mathrm{d}t}\log\rho\right) . \end{aligned} \tag{17}$$

The variation of ρ can be computed using perturbation theory [1]:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -i\,\mathrm{tr}_{BB'}[H, |\psi\rangle\langle\psi|] = 2\,\mathrm{Im}\,\mathrm{tr}_{BB'}(H|\psi\rangle\langle\psi|) . \tag{18}$$

Letting $R = U\sqrt{\Lambda}V^T$ and considering $H = K_{xx}$, we have

$$\begin{aligned} &\mathrm{tr}_{BB'}\,(K_{xx}|\psi\rangle\langle\psi|) \\ &= \mathrm{tr}_{BB'}\left[(X\otimes X)(R\otimes I_{BB'})|\Phi\rangle\langle\Phi|(R^\dagger \otimes I_{BB'})\right] \\ &= \mathrm{tr}_{BB'}\left[(XRX^T \otimes I_{BB'})|\Phi\rangle\langle\Phi|(R^\dagger\otimes I_{BB'})\right] \\ &= XRX^TR^\dagger , \end{aligned} \tag{19}$$

where we have introduced $X := \sigma_x \otimes I$, with the identity operator acting on the ancilla. The first equality follows simply from substitution of K_{xx} and $|\psi\rangle$ by their expressions in Eqs. (11) and (15); the second uses Eq. (16); and the third employs the fact that for any operators M_1, M_2,

$$\mathrm{tr}_{BB'}[(M_1\otimes I_{BB'})|\Phi\rangle\langle\Phi|(M_2\otimes I_{BB'})] = M_1M_2 . \tag{20}$$

Since $\rho = U\Lambda U^\dagger$, we have

$$E_{K_{xx}} = \sup_{|\psi\rangle}\mathrm{tr}\left(-U^\dagger\frac{\mathrm{d}\rho}{\mathrm{d}t}U\log\Lambda\right) . \tag{21}$$

Using Eqs. (18) and (19), and introducing the Hermitian operators $X_U = U^\dagger X U$ and $X_V = V^\dagger X V$, we have

$$U^\dagger\frac{\mathrm{d}\rho}{\mathrm{d}t}U = 2\,\mathrm{Im}\,X_U\sqrt{\Lambda}X_V^T\sqrt{\Lambda} . \tag{22}$$

Letting U, V, Λ attain the supremum in Eq. (21) (up to an amount that can be made arbitrarily small), we find

$$\begin{aligned} E_{K_{xx}} &= -2\,\mathrm{Im}\,\mathrm{tr}\left(X_U\sqrt{\Lambda}X_V^T\sqrt{\Lambda}\log\Lambda\right) \\ &= i\,\mathrm{tr}\left[(X_U\sqrt{\Lambda}X_V^T - X_V^T\sqrt{\Lambda}X_U)\sqrt{\Lambda}\log\Lambda\right] \\ &= i\,\mathrm{tr}\left[M(X_U\circ X_V)\right] , \end{aligned} \tag{23}$$

where we have introduced the real, skew-symmetric matrix

$$M_{ij} := \sqrt{\lambda_i\lambda_j}\log(\lambda_j/\lambda_i) , \tag{24}$$

and the symbol $\circ$ denotes the Hadamard (i.e., element-wise) product of matrices. In the second line of Eq. (23) we have used

$$\mathrm{Im}\,\mathrm{tr}\,A = \mathrm{tr}(A - A^\dagger)/2i \tag{25}$$

and the fact that Λ, X_U, and X_V are Hermitian. The last line can be checked by explicitly writing the trace in terms of matrix elements.

From Eq. (23) we obtain the following upper bound for $E_{K_{xx}}$ (here $\wr A\wr$ denotes the element-wise absolute value, i.e., $\wr A\wr_{ij} = |A_{ij}|$):

$$\begin{aligned} E_{K_{xx}} &\le \mathrm{tr}(\wr M\wr\,\wr X_U\circ X_V\wr) \\ &\le \sup_P \mathrm{tr}(\wr M\wr P) \\ &\le 2\max_x \sqrt{x(1-x)}\log[x/(1-x)] \\ &= \alpha \approx 1.9123 , \end{aligned} \tag{26}$$

where P is a permutation operator and $x \in [0,1]$. The first line uses the triangle inequality. The second inequality follows from noticing that $\wr X_U\circ X_V\wr$ is a doubly substochastic matrix [16]. Indeed, for any two complex numbers v and w one has that $2|vw| \le |v|^2 + |w|^2$, and consequently, for any two unitary matrices V and W,

$$\begin{aligned} \sum_i |V_{ij}W_{ij}| \le \sum_i(|V_{ij}|^2 + |W_{ij}|^2)/2 = 1, \\ \sum_j |V_{ij}W_{ij}| \le \sum_j(|V_{ij}|^2 + |W_{ij}|^2)/2 = 1, \end{aligned} \tag{27}$$

which implies that the matrix $\wr V\circ W\wr$, with entries $|V_{ij}W_{ij}|$, is doubly substochastic. Therefore a doubly stochastic matrix Q exists such that $|X_U\circ X_V|_{ij} \le Q_{ij}$ for all i and j [16], so that $\mathrm{tr}(\wr M\wr\,\wr X_U\circ X_V\wr) \le \mathrm{tr}(\wr M\wr Q)$. But Q is a convex combination of permutation operators P_k, $Q = \sum_k p_k P_k$, which implies that $\mathrm{tr}\,\wr M\wr Q \le \sup_P \mathrm{tr}(\wr M\wr P)$. Finally, the third inequality in Eq. (26) follows from noticing that

$$\begin{aligned} |M|_{ij} &= \sqrt{\lambda_i\lambda_j}|\log(\lambda_j/\lambda_i)| \\ &= (\lambda_i+\lambda_j)\sqrt{\frac{\lambda_i}{\lambda_i+\lambda_j}\frac{\lambda_j}{\lambda_i+\lambda_j}}|\log(\lambda_j/\lambda_i)| \\ &\le (\lambda_i+\lambda_j)\max_x\sqrt{x(1-x)}\log[x/(1-x)] \\ &= (\lambda_i+\lambda_j)\alpha/2 , \end{aligned} \tag{28}$$

and that

$$\mathrm{tr}(\wr M\wr P) \le \frac{\alpha}{2}\sum_{ij}(\lambda_i+\lambda_j)P_{ij} = \alpha\sum_i\lambda_i = \alpha , \tag{29}$$

where we have used the facts that P is a permutation matrix and that $\sum_i \lambda_i = 1$. Comparison of Eqs. (7) and (26) shows that, indeed, $E_{K_{xx}} = E_{K_{xx}}^{(1*)}$, completing the proof.

We have shown that ancillary systems are not needed when optimizing entanglement generation by any two-qubit Hamiltonian with canonical form given by Eq. (9). More specifically, there is a universal optimal two-qubit initial state given by [1]

$$|\psi_{\max}\rangle := \sqrt{x_0}|0\rangle_A \otimes |1\rangle_B - i\sqrt{1-x_0}|1\rangle_A \otimes |0\rangle_B \,. \quad (30)$$

As an application of the above, we discuss how to use the Hamiltonian K to enable classical communication between Alice and Bob. This has been studied in [2], in which the entanglement assisted forward classical capacity $C^E_{\rightarrow}$ (the maximum rate for the Hamiltonian H to communicate from Alice to Bob when free, unlimited shared entanglement is available) is shown to be

$$C^E_{\rightarrow}(H) = \sup_{\mathcal{E}} \left[\lim_{t\to 0} \frac{\chi(\mathrm{tr}_{AA'}\, e^{-iHt}\mathcal{E}) - \chi(\mathrm{tr}_{AA'}\,\mathcal{E})}{t} \right], \quad (31)$$

where $\mathcal{E} = \{p_i, |\psi_i\rangle\}$ is an ensemble of bipartite states, $e^{-iHt}\mathcal{E}$ and $\mathrm{tr}_{AA'}\,\mathcal{E}$ denote the respective transformed ensembles $\{p_i, e^{-iHt}|\psi_i\rangle\}$ and $\{p_i, \mathrm{tr}_{AA'}\,|\psi_i\rangle\langle\psi_i|\}$, and

$$\chi(\{p_i,\rho_i\}) := S\left(\sum_i p_i\rho_i\right) - \sum_i p_i S(\rho_i) \quad (32)$$

is the Holevo information of the ensemble $\{p_i, \rho_i\}$, where S is the von Neumann entropy. Reference [2] also describes a protocol to achieve the rate in the bracket of Eq. (31) for any ensemble $\mathcal{E}$.

For any two-qubit Hamiltonian H, Ref. [6] constructs an ensemble with communication rate E_H, which implies $C^E_{\rightarrow}(H) \geq E_H$. This ensemble, which is not necessarily optimal, is defined in terms of an optimal state for entanglement generation. This ensemble $\mathcal{E}_1$ can now be made more explicit for Hamiltonian K in light of our findings:

$$\begin{aligned} p_1 &:= \frac{1}{2},\ |\psi_1\rangle := \sqrt{x_0}|0\rangle_A \otimes |1\rangle_B + i\sqrt{1-x_0}|1\rangle_A \otimes |0\rangle_B, \\ p_2 &:= \frac{1}{2},\ |\psi_2\rangle := \sqrt{x_0}|0\rangle_A \otimes |0\rangle_B - i\sqrt{1-x_0}|1\rangle_A \otimes |1\rangle_B, \end{aligned}$$

where x_0 is defined after Eq. (7). For ensemble $\mathcal{E}_1$ we find

$$\begin{aligned} \chi(\mathrm{tr}_A\,\mathcal{E}_1) &= S(I/2) - S(\mathrm{tr}_A\,|\psi_1\rangle\langle\psi_1|) = 1 - E(|\psi_{\max}\rangle) \\ \chi(\mathrm{tr}_A(e^{-i\delta t K}\mathcal{E}_1)) &= 1 - [E(|\psi_{\max}\rangle) - \delta t E_K] \end{aligned} \quad (33)$$

and therefore the net rate at which classical bits are transmitted is indeed $\Delta\chi/\delta t = E_K$.

Next we present an alternative ensemble $\mathcal{E}_2$ of product states with the same communication rate:

$$\begin{aligned} p_1 &:= \frac{1}{2},\ |\psi_1\rangle := \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}_A \otimes \begin{pmatrix}\sqrt{x_0}\\ -i\sqrt{1-x_0}\end{pmatrix}_B \\ p_2 &:= \frac{1}{2},\ |\psi_2\rangle := \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}_A \otimes \begin{pmatrix}\sqrt{x_0}\\ i\sqrt{1-x_0}\end{pmatrix}_B . \end{aligned}$$

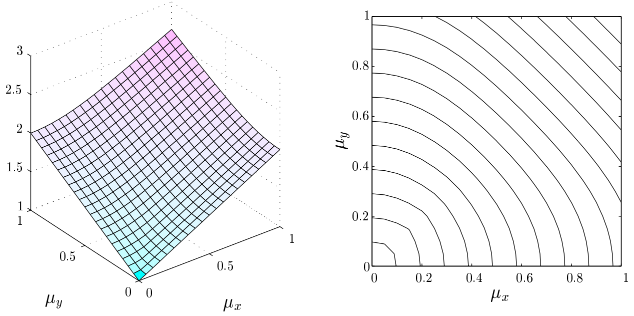

FIG. 1: Numerically optimized entanglement capacity of the two-qubit Hamiltonian $\mu_x\ \sigma_x\otimes\sigma_x + \mu_y\ \sigma_y\otimes\sigma_y + \sigma_z\otimes\sigma_z$ with single qubit ancillas on each side. The vertical axis in the left figure is in units of α.

Here, we use K to simulate K' [9], under which the ensemble evolves. For ensemble $\mathcal{E}_2$, $S(\mathrm{tr}_A\,|\psi_i\rangle\langle\psi_i|) = 0$, so

$$\begin{aligned} \chi(\mathrm{tr}_A\,\mathcal{E}_2) &= H_2(x_0) \\ \chi(\mathrm{tr}_A(e^{-i\delta t K}\mathcal{E}_2)) &= H_2\left(x_0 - 2\delta t\sqrt{x_0(1-x_0)}\right) \\ &= H_2(x_0) + E_K\delta t \end{aligned} \quad (34)$$

(where H_2 is the binary entropy). Thus the communication rate is again $\Delta\chi/\delta t = E_K$.

The main difference between these two ensembles is that the states in ensemble $\mathcal{E}_1$ are entangled whereas the states in ensemble $\mathcal{E}_2$ are not. In the first case the interaction K is used to decrease the degree of entanglement between Alice and Bob or, equivalently, to make the states of Bob's ensemble $\mathrm{tr}_A\,\mathcal{E}_1$ less mixed and thus more distinguishable. The same increase of distinguishability for the pure states of Bob's ensemble $\mathrm{tr}_A\,\mathcal{E}_2$ is achieved by conditionally rotating them with K, in a way that they become more orthogonal to each other. We note, in addition, that ensembles $\mathcal{E}_1$ and $\mathcal{E}_2$ can be prepared using different remote state preparation techniques [17].

In conclusion, we have computed the asymptotic entanglement capacities of all two-qubit Hamiltonians that are locally equivalent to $\mu_x\ \sigma_x\otimes\sigma_x + \mu_y\ \sigma_y\otimes\sigma_y$ by showing that this capacity can be achieved without the use of ancillas. However, as discussed above, ancillas are necessary to achieve the capacity in general. Although we do not have a closed form expression for the capacity of an arbitrary two-qubit Hamiltonian, we can present partial results in this direction. The numerically optimized entanglement capacity of a general two-qubit Hamiltonian is shown in Fig. 1. Numerically, we find that the optimum can be achieved with single-qubit ancillas on both sides. For Hamiltonians of the form $K_{\mu_{xy}} = \mu_{xy}(\sigma_x\otimes\sigma_x + \sigma_y\otimes\sigma_y) + \sigma_z\otimes\sigma_z$, we conjecture that the entanglement capacity is given by

$$\begin{aligned} E_{K_{\mu_{xy}}} = 2\max\big\{ &\sqrt{p_1p_2}\log(p_1/p_2)\,[\sin\theta + \mu_{xy}\sin(\varphi-\xi)] \\ &+ \sqrt{p_2p_4}\log(p_2/p_4)\,[\sin\varphi + \mu_{xy}\sin(\theta-\xi)] \\ &+ \sqrt{p_1p_4}\log(p_1/p_4)\,\mu_{xy}\sin\xi \big\} \end{aligned} \quad (35)$$

where the maximum is taken over $p_1 > 0$, $p_2 > 0$, $p_4 = 1 - p_1 - 2p_2 > 0$, and $\theta, \varphi, \xi \in [0, 2\pi)$. This expression was found by investigating the structure of the numerical optimum, and it agrees well with the numerical results. It does not seem possible to simplify this expression further, which suggests that in general, capacities may not have simple closed form expressions, but can only be expressed as maximizations of multivariable transcendental functions. Nevertheless, it would be useful to show that this maximization can be taken over a finite number of parameters by proving an upper bound on the dimension of the ancillas.

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